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Three-point spectral functions in ϕ_6^3 theory at finite temperature

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Abstract. We derive a set of relations among the thermal components of the 3-point function and its spectral representations at finite temperature in the real-time formalism. We then use these to calculate, in certain kinematic limits, the 3-point spectral densities for ϕ_6^3 theory and relate the result to the case of hot QCD.

I Introduction

Spectral functions are essential and useful in finite temperature field theory [1,2] because a large number of transport coefficients are given directly by them [3–5]. Furthermore, studying the spectral functions may help us to understand the quasi-particle structure of field theories at finite temperature as well as to identify the microscopic processes underlying their dynamics. In this paper we derive expressions for the spectral densities of the 3-point Green functions in finite temperature field theories within the Closed Time Path (CTP) formalism [6–8] and evaluate the spectral densities of the 3-point function for ϕ^3 theory using resummed propagators in the "hard thermal loop" (HTL) approximation [9–11].

In the imaginary-time formalism (ITF) [2], one obtains the spectral densities from the discontinuity of the Green functions across the real energy axis after performing an analytic continuation of the imaginary external energy variables to the real axis [12]. Explicit expressions for the spectral densities for the three gluon ITF vertex in QCD in HTL approximation were derived in [11].

In real-time formulations of finite temperature field theory the number of degrees of freedom is doubled, leading to a 2×2 matrix structure of the single particle propagators. The external energies remain real, and the complicated summation over the Masubara frequencies followed by analytic continuation is avoided. In [13] Kobes and Semenoff derived Cutkosky rules for calculating the imaginary parts of thermal two-point functions using the formalism of Thermo-Field Dynamics (TFD). Spectral representations of the 3-point Green functions were derived in [14] using the notation of "circled" vertices. Recently these cutting rules were reexamined in the CTP formalism in [15,16] and given a simple physical interpretation in [17].

A useful technical simplification for perturbative calculations in real-time finite temperature field theory is provided by the decomposition and spectral representation of the 3-point vertex given in [18]. Missing in that paper is an explicit expression of the spectral densities in terms of the thermal components of the real-time 3-point vertex function. This hole is filled in by the present paper.

We then apply these expressions to the 3-point vertex for ϕ^3 theory in 6 dimensions. We study the corresponding spectral densities in the 1-loop approximation for soft external momenta. Explicit results are given for vanishing external spatial momenta. It is well-known [2,9] that field theories with massless degrees of freedom develop at non-zero temperature infrared divergences which usually signal dynamical mass generation and in many cases can be dealt with by resummation of the "hard thermal loops" [9]. In ϕ^3 theory the situation is even a little more complicated: the effective potential is unbounded from below, and in the massless limit the theory doesn't even have a metastable ground state. By adding to the Lagrangean a non-zero, positive mass term the theory develops a metastable, local minimum at $\langle\phi\rangle = 0$ which, at zero temperature, is perturbatively stable in the limit of small 3-point coupling constant g [10]. At non-zero temperature, however, the tadpole diagram contains a temperature dependent finite, but negative contribution which shifts the position $\langle\phi\rangle$ of the metastable vacuum to negative values and reduces the effective boson mass [10]. This effect must be taken into account self-consistently via resummed (massive) propagators in order to avoid an expansion around the wrong vacuum.

We will use the CTP formalism [6,7] throughout this paper in the form given in [8,18]. In this representation of the real-time formalism the single-particle propagator in momentum space has the form

$$D(p) = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{1}$$

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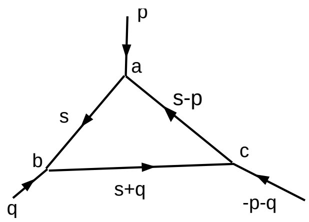

Fig. 1. 3-point vertex in ϕ^3 theory

where

$$i\,D_{11}(p) = (i\,D_{22})^* = i\mathcal{P}\left(\frac{1}{p^2-m^2}\right) + \left(n(p_0)+\frac{1}{2}\right)\rho(p)\,, \quad (2a)$$

$$i\,D_{12}(p) = n(p_0)\,\rho(p)\,, \quad (2b)$$

$$i\,D_{21}(p) = \big(1+n(p_0)\big)\,\rho(p)\,. \quad (2c)$$

Here $n(p_0)$ is the thermal Bose-Einstein distribution

$$n(p_0) = \frac{1}{e^{\beta p_0}-1}\,, \quad (3)$$

and $\rho(p)$ is the two-point spectral density which for free particles is given by

$$\rho(p) = 2\pi\,\mathrm{sgn}(p_0)\,\delta(p^2-m^2)\,. \quad (4)$$

The paper is organized as follows. In Sect. II we review some useful general relations among the thermal components of the 3-point function and their spectral representations, both for the connected and for the truncated vertices. In Sect. III we evaluate the spectral densities for the truncated 3-point vertex in ϕ^3 in 1-loop approximation. In Sect. IV we discuss and summarize our results. Some technical details of the calculations and further useful relations are given in the Appendix.

II Spectral representation of the 3-point vertex

In this section we shortly review some useful relations among the different thermal components of the 3-point functions and their spectral representations. Equivalent (although not identical) relations have been reported in the literature [7,13,14,19,20] in different notation. For simplicity of presentation we consider the 3-point vertex function for ϕ^3 theory, see Fig. 1. The three incoming external momenta are $k_1 = p$, $k_2 = q$, and $k_3 = -p-q$.

II.1 Relations among the thermal components of the real-time vertex

The thermal components of the connected 3-point vertex function are defined by [7]

$$\Gamma_{111} = \langle T(\phi_1\phi_2\phi_3)\rangle\,, \quad (5a)$$

$$\Gamma_{112} = \langle \phi_3 T(\phi_1\phi_2)\rangle\,, \quad (5b)$$

$$\Gamma_{121} = \langle \phi_2 T(\phi_1\phi_3)\rangle\,, \quad (5c)$$

$$\Gamma_{211} = \langle \phi_1 T(\phi_2\phi_3)\rangle\,, \quad (5d)$$

$$\Gamma_{122} = \langle \tilde{T}(\phi_2\phi_3)\phi_1\rangle\,, \quad (5e)$$

$$\Gamma_{212} = \langle \tilde{T}(\phi_1\phi_3)\phi_2\rangle\,, \quad (5f)$$

$$\Gamma_{221} = \langle \tilde{T}(\phi_1\phi_2)\phi_3\rangle\,, \quad (5g)$$

$$\Gamma_{222} = \langle \tilde{T}(\phi_1\phi_2\phi_3)\rangle\,, \quad (5h)$$

where $\phi_1 \equiv \phi(x_1) = \phi(\boldsymbol{x}_1, t_1)$ etc., and $\Gamma_{abc} \equiv \Gamma_{abc}(x_1,x_2,x_3)$). Following [21] we defined the process of "tilde conjugation" by reversing the time order in coordinate space: time-ordered products become products with anti-chronological ordering, and $\theta(t)$ becomes $\theta(-t)$.

Using the identity $\theta(t)+\theta(-t)=1$ it is straightforward to show that

$$\sum_{a,b,c=1}^{2} (-1)^{a+b+c-3}\Gamma_{abc} = 0\,. \quad (6)$$

In momentum space tilde conjugation turns out to be equivalent to complex conjugation and, using the KMS condition, one finds [7,20]

$$\tilde{\Gamma}_{111}(k_1,k_2,k_3) = \Gamma^*_{111}(k_1,k_2,k_3) = \Gamma_{222}(k_1,k_2,k_3)\,, \quad (7a)$$

$$\tilde{\Gamma}_{121}(k_1,k_2,k_3) = \Gamma^*_{121}(k_1,k_2,k_3) = e^{\beta\omega_2}\,\Gamma_{212}(k_1,k_2,k_3)\,, \quad (7b)$$

$$\tilde{\Gamma}_{211}(k_1,k_2,k_3) = \Gamma^*_{211}(k_1,k_2,k_3) = e^{\beta\omega_1}\,\Gamma_{122}(k_1,k_2,k_3)\,, \quad (7c)$$

$$\tilde{\Gamma}_{112}(k_1,k_2,k_3) = \Gamma^*_{112}(k_1,k_2,k_3) = e^{\beta\omega_3}\,\Gamma_{221}(k_1,k_2,k_3)\,, \quad (7d)$$

where $k_i = (\omega_i, \boldsymbol{k}_i)$ and $k_1+k_2+k_3=0$. These identities show that at most three of the eight thermal components of the real-time vertex function are independent.

II.2 Largest and smallest time equations

If t_3 is the largest time argument, one obtains from (5a) and (5b) the identities

$$\theta_{32}\,\theta_{21}\,\Gamma_{111} = \langle\phi_3\phi_2\phi_1\rangle = \theta_{32}\,\theta_{21}\,\Gamma_{112} \quad (8)$$

or

$$\theta_{32}\,\theta_{21}\,(\Gamma_{111}-\Gamma_{112}) = 0\,. \quad (9)$$

Here $\theta_{ij} \equiv \theta(t_i - t_j)$. Similarly one derives the more general relations

$$\theta_{32}\,\theta_{21}\,(\Gamma_{ab1}-\Gamma_{ab2}) = 0 = \theta_{31}\,\theta_{12}\,(\Gamma_{ab1}-\Gamma_{ab2})\,, \quad (10a)$$

$$\theta_{21}\,\theta_{13}\,(\Gamma_{a1b}-\Gamma_{a2b}) = 0 = \theta_{23}\,\theta_{31}\,(\Gamma_{a1b}-\Gamma_{a2b})\,, \quad (10b)$$

$$\theta_{13}\,\theta_{32}\,(\Gamma_{1ab}-\Gamma_{2ab}) = 0 = \theta_{12}\,\theta_{23}\,(\Gamma_{1ab}-\Gamma_{2ab})\,, \quad (10c)$$

where a and b can be either 1 or 2. By tilde conjugation one obtains from these equations the following relations:

$$\theta_{12}\,\theta_{23}\,(\tilde{\Gamma}_{ab1} - \tilde{\Gamma}_{ab2}) = 0 = \theta_{21}\,\theta_{13}\,(\tilde{\Gamma}_{ab1} - \tilde{\Gamma}_{ab2})\,, \quad (11a)$$

$$\theta_{31}\,\theta_{12}\,(\tilde{\Gamma}_{a1b} - \tilde{\Gamma}_{a2b}) = 0 = \theta_{13}\,\theta_{32}\,(\tilde{\Gamma}_{a1b} - \tilde{\Gamma}_{a2b})\,, \quad (11b)$$

$$\theta_{23}\,\theta_{31}\,(\tilde{\Gamma}_{1ab} - \tilde{\Gamma}_{2ab}) = 0 = \theta_{32}\,\theta_{21}\,(\tilde{\Gamma}_{1ab} - \tilde{\Gamma}_{2ab})\,, \quad (11c)$$

Equations (10) and (11) are the analogues of the "largest time equations" and "smallest time equations", respectively, of [13]. They will be used extensively in the derivation of the spectral representations of Appendix A. Their generalization to arbitrary n-point functions is straightforward.

II.3 Physical vertex functions

One can construct the "retarded", "forward", and "even" vertex functions from the eight components of the 3-point function as [7,18].

$$\Gamma_R = \Gamma_{111} - \Gamma_{112} - \Gamma_{211} + \Gamma_{212}, \quad (12a)$$
$$\Gamma_{Ri} = \Gamma_{111} - \Gamma_{112} - \Gamma_{121} + \Gamma_{122}, \quad (12b)$$
$$\Gamma_{Ro} = \Gamma_{111} - \Gamma_{121} - \Gamma_{211} + \Gamma_{221}, \quad (12c)$$
$$\Gamma_{F} = \Gamma_{111} - \Gamma_{121} + \Gamma_{212} - \Gamma_{222}, \quad (12d)$$
$$\Gamma_{Fi} = \Gamma_{111} + \Gamma_{122} - \Gamma_{211} - \Gamma_{222}, \quad (12e)$$
$$\Gamma_{Fo} = \Gamma_{111} - \Gamma_{112} + \Gamma_{221} - \Gamma_{222}, \quad (12f)$$
$$\Gamma_{E} = \Gamma_{111} + \Gamma_{122} + \Gamma_{212} + \Gamma_{221}, \quad (12g)$$

Inversion of these equations together with (6) yields expressions for the thermal components Γ_{abc} in terms the above "physical" vertex functions; they are given in compact form in [18].

Using (6) and (7) one can eliminate the "forward" and "even" vertex functions in terms of the three retarded vertices. Thus all components of Γ_{abc} can be expressed through Γ_R, Γ_{Ri}, and Γ_{Ro} [18,19].

II.4 Spectral integral representations

In [18] the following integral representations for the retarded vertex functions in momentum space were derived (in slightly different notation):

$$\Gamma_R(\omega_1,\omega_2,\omega_3) = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\Omega_1 d\Omega_2}{\omega_2-\Omega_2+i\epsilon} \times\left(\frac{\rho_1}{\omega_1-\Omega_1-i\epsilon}+\frac{\rho_1-\rho_2}{\omega_3-\Omega_3-i\epsilon}\right), \quad (13a)$$

$$\Gamma_{Ri}(\omega_1,\omega_2,\omega_3) = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\Omega_1 d\Omega_2}{\omega_1-\Omega_1+i\epsilon} \times\left(\frac{\rho_2}{\omega_2-\Omega_2-i\epsilon}-\frac{\rho_1-\rho_2}{\omega_3-\Omega_3-i\epsilon}\right), \quad (13b)$$

$$\Gamma_{Ro}(\omega_1,\omega_2,\omega_3) = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\Omega_1 d\Omega_2}{\omega_3-\Omega_3+i\epsilon} \times\left(\frac{\rho_1}{\omega_1-\Omega_1-i\epsilon}+\frac{\rho_2}{\omega_2-\Omega_2-i\epsilon}\right). \quad (13c)$$

The spatial momenta $\boldsymbol{k}_1$, $\boldsymbol{k}_2$, $\boldsymbol{k}_3{=}-(\boldsymbol{k}_1{+}\boldsymbol{k}_2)$ are the same on both sides of these equations and have therefore been suppressed. The frequency arguments of the spectral functions under the integrals are $\rho_i \equiv \rho_i(\Omega_1,\Omega_2,\Omega_3)$, with $\Omega_1+\Omega_2+\Omega_3=0$. In Appendix A we give a short derivation of these integral representations from which it follows that in momentum space

$$\rho_1 = \mathrm{Im}\,(\Gamma_{122}+\Gamma_{211})\,, \quad (14a)$$
$$\rho_2 = \mathrm{Im}\,(\Gamma_{121}+\Gamma_{212})\,. \quad (14b)$$

The spectral integral representations (13) differ from those given in (31) of [18] because they use different spectral densities. The spectral functions ρ_1, ρ_2 used here are not simply related to ρ_A, ρ_B of [18]: while it follows from (14) that ρ_1 and ρ_2 are real in momentum space, ρ_A and ρ_B are instead real in coordinate space and satisfy a more complicated relation ((28) of [18]) in momentum space. Still, both sets of spectral integral representations are correct; the one given here appears to simplify things in practice, however (see below).

Similar spectral representations can be derived for the truncated (1-particle irreducible, 1PI) vertex functions. The technical steps are given in Appendix A.2, together with the corresponding generalizations to 1PI vertex functions for the relations derived in the preceding subsections. Here we only state the result:

$$G_R(\omega_1,\omega_2,\omega_3) = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\Omega_1 d\Omega_2}{\omega_2-\Omega_2+i\epsilon} \times\left(\frac{\rho'_1}{\omega_1-\Omega_1-i\epsilon}+\frac{\rho'_1-\rho'_2}{\omega_3-\Omega_3-i\epsilon}\right), \quad (15a)$$

$$G_{Ri}(\omega_1,\omega_2,\omega_3) = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\Omega_1 d\Omega_2}{\omega_1-\Omega_1+i\epsilon} \times\left(\frac{\rho'_2}{\omega_2-\Omega_2-i\epsilon}-\frac{\rho'_1-\rho'_2}{\omega_3-\Omega_3-i\epsilon}\right), \quad (15b)$$

$$G_{Ro}(\omega_1,\omega_2,\omega_3) = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty}\frac{d\Omega_1 d\Omega_2}{\omega_3-\Omega_3+i\epsilon} \times\left(\frac{\rho'_1}{\omega_1-\Omega_1-i\epsilon}+\frac{\rho'_2}{\omega_2-\Omega_2-i\epsilon}\right), \quad (15c)$$

with

$$\rho'_1 = \mathrm{Im}\,(G_{122}-G_{211})\,, \quad (16a)$$
$$\rho'_2 = \mathrm{Im}\,(G_{212}-G_{212})\,. \quad (16b)$$

These results should be compared with the expressions derived by Kobes in [14] which have a similar structure but use three somewhat differently defined spectral densities.

III 1-Loop spectral densities for the vertex in ϕ^3_6 theory

In this section we calculate the 1-loop contribution to the spectral functions ρ'_1, ρ'_2 for the 1PI 3-point vertex in ϕ^3 theory. As interaction term in the Lagrangean we use $\frac{g}{6}\phi^3$. From (16) and (A26) we have

$$\rho'_1 = \mathrm{Im}\,(G_{122} + e^{\beta p_0} G^*_{122}) = -\frac{1}{n(p_0)} \mathrm{Im}\, G_{122}\,. \qquad (17)$$

For ρ'_1 we thus must evaluate only the single Feynman diagram in Fig. 1 for $a = 1$, $b = c = 2$. Using standard real-time Feynman rules [7,8] one gets (in n space-time dimensions)

$$\begin{aligned} &G_{122}(p, q, -p-q) \\ &= (-ig)(ig)^2 \int \frac{d^n s}{(2\pi)^n} \\ &\quad \times [iD_{12}(s)][iD_{22}(s+q)][iD_{21}(s-p)]\,. \end{aligned} \qquad (18)$$

Inserting the thermal free propagators (2), extracting the imaginary part, and performing the integration over s^0 with the help of the function $\delta(s^2 - m^2) = [\delta(s_0 - E_s) + \delta(s_0 + E_s)]/2E_s$, where $E_s = \sqrt{m^2 + \boldsymbol{s}^2}$, one finds

$$\begin{aligned} &\rho_1(p, q, -p-q) \\ &= -\frac{g^3}{n(p_0)} \Big(A(p,q) + B(p,q) \Big)\,, \end{aligned} \qquad (19a)$$

$$\begin{aligned} &A(p,q) \\ &= \int \frac{d^{n-1}s}{(2\pi)^{n-3}} \frac{1}{2E_s} \mathrm{sgn}(E_s + q_0)\, \mathrm{sgn}(E_s - p_0) \\ &\quad \times \delta\big((E_s + q_0)^2 - E^2_{s+q}\big)\, \delta\big((E_s - p_0)^2 - E^2_{s-p}\big) \\ &\quad \times n(E_s) \left(\tfrac{1}{2} + n(E_s + q_0)\right) \big(1 + n(E_s - p_0)\big)\,, \end{aligned} \qquad (19b)$$

$$\begin{aligned} &B(p,q) \\ &= \int \frac{d^{n-1}s}{(2\pi)^{n-3}} \frac{1}{2E_s} \mathrm{sgn}(E_s - q_0)\, \mathrm{sgn}(E_s + p_0) \\ &\quad \times \delta\big((E_s - q_0)^2 - E^2_{s+q}\big)\, \delta\big((E_s + p_0)^2 - E^2_{s-p}\big) \\ &\quad \times \big(1 + n(E_s)\big) \left(\tfrac{1}{2} + n(E_s - q_0)\right) n(E_s + p_0)\,. \end{aligned} \qquad (19c)$$

Here $E_{s+q} = \sqrt{m^2 + (\boldsymbol{s}+\boldsymbol{q})^2}$, $E_{s-p} = \sqrt{m^2 + (\boldsymbol{s}-\boldsymbol{p})^2}$, and in (19c) we used the identity $n(-x) = -\big(1 + n(x)\big)$.

The integrands in (19b,c) contain up to three powers of the thermal Bose distribution functions. Superficial power counting thus suggests severe infrared singularities in the massless limit. On the other hand, 1-loop integrals in the imaginary time formalism are always linear in the thermal distribution functions which arise from conversion of a single sum over discrete loop frequencies into a complex contour integral [9]. This suggests that the infrared problems resulting from higher powers of the distribution functions in the real time formalism are spurious. In fact, it was already noted in [22,23] that the cubic terms cancel from the retarded 3-point functions. Using the identity

$$n(a)\, n(b) = n(a+b) \big(1 + n(a) + n(b)\big) \qquad (20)$$

one can show that also the quadratic terms disappear, and that A and B reduce to

$$\begin{aligned} &A(p,q) \\ &= n(p_0) \int \frac{d^{n-1}s}{(2\pi)^{n-3}} \frac{1}{2E_s} \mathrm{sgn}(E_s + q_0)\, \mathrm{sgn}(E_s - p_0) \\ &\quad \times \delta\big((E_s + q_0)^2 - E^2_{s+q}\big)\, \delta\big((E_s - p_0)^2 - E^2_{s-p}\big) \\ &\quad \times \Big[\tfrac{1}{2} \big(n(E_s - p_0) - n(E_s)\big) \\ &\qquad + n(p_0 + q_0) \big(n(E_s - p_0) - n(E_s + q_0)\big) \\ &\qquad + n(q_0) \big(n(E_s + q_0) - n(E_s)\big) \Big]\,, \end{aligned} \qquad (21a)$$

$$\begin{aligned} &B(p,q) \\ &= n(p_0) \int \frac{d^{n-1}s}{(2\pi)^{n-3}} \frac{1}{2E_s} \mathrm{sgn}(E_s - q_0)\, \mathrm{sgn}(E_s + p_0) \\ &\quad \times \delta\big((E_s - q_0)^2 - E^2_{s+q}\big)\, \delta\big((E_s + p_0)^2 - E^2_{s-p}\big) \\ &\quad \times \Big[\tfrac{1}{2} \big(n(E_s + p_0) - n(E_s)\big) \\ &\qquad + n(p_0 + q_0) \big(n(E_s + p_0) - n(E_s - q_0)\big) \\ &\qquad + n(q_0) \big(n(E_s - q_0) - n(E_s)\big) \Big]\,. \end{aligned} \qquad (21b)$$

Note that the factor $n(p_0)$ in front of the integrals cancels the distribution function in the denominator of (17), (19a). The remaining integrands are linear in the thermal distribution functions $n(E_s \pm \ldots)$ and are infrared finite even in the massless limit [24].

To simplify the notation it is convenient to introduce the 4-vectors $V_\pm = \left(1, \pm \frac{\boldsymbol{s}}{E_s}\right)$. With their help we can rewrite the arguments of the δ-functions in (19) as

$$(s+q)^2\big|_{s^0 = \pm E_s} = m^2 + q^2 \pm 2E_s\, q \cdot V_\pm\,, \qquad (22a)$$

$$(s-p)^2\big|_{s^0 = \pm E_s} = m^2 + p^2 \mp 2E_s\, p \cdot V_\pm\,. \qquad (22b)$$

We will consider the theory in the weak coupling limit, $g \ll 1$. As shown in [10], for $n = 6$ the theory becomes perturbatively unstable[1] for temperatures above

$$T_{\rm cr} = \left(\frac{180}{\pi}\right)^{1/4} \frac{m}{\sqrt{g}}\,, \qquad (23)$$

so for given T we must use massive propagators with a mass that satisfies $m > \sqrt{g}\,T$. We will consider the case $\sqrt{g}\,T \lesssim m \ll T$ and calculate the spectral density for the vertex for soft external momenta, q, $p \sim m \sim \sqrt{g}\,T \ll T$.

For gauge theories it is known that in the limit of soft external momenta there is a "hard thermal loop" (HTL) contribution to the 3-gluon vertex which is of the same order in the coupling constant as the tree-level result and must therefore be resummed in a complete leading order calculation [9]. To see whether such a resummation is also required in scalar ϕ^3 theory we first evaluate the functions

[1] Stability problems for perturbation theory in massless ϕ^3 theory were recently also discussed in the context of relativistic transport theory in [25].

A, B in the hard thermal loop approximation, by assuming that the loop integral in (22) is dominated by "hard" momenta $\bar{s} = |\boldsymbol{s}| \sim T$ [9]. For such momenta we can neglect the rest mass m, $E_s \approx \bar{s}$, and the arguments of the δ-functions can be approximated according to

$$(s+q)^2\big|_{s^0=\pm E_s} \approx \pm 2E_s\, q\cdot V_\pm\,, \tag{24a}$$

$$(s-p)^2\big|_{s^0=\pm E_s} \approx \mp 2E_s\, p\cdot V_\pm\,. \tag{24b}$$

Setting further

$$\mathrm{sgn}(E_s \pm p^0) \approx 1 \approx \mathrm{sgn}(E_s \pm q^0)\,, \tag{25a}$$

$$q\cdot V_\pm \approx q^0 \mp |\boldsymbol{q}|\cos\theta'\,, \tag{25b}$$

$$p\cdot V_\pm \approx p^0 \mp |\boldsymbol{p}|\cos\theta\,, \tag{25c}$$

where θ, θ' are the angles between $\boldsymbol{s}$ and $\boldsymbol{p}$, $\boldsymbol{q}$, respectively, the angular and radial integrations in (21) decouple [9]. We thus obtain

$$A(p,q) + B(p,q)\Big|_{\mathrm{HTL}} = n(p_0)\, a(p_0,q_0)\, \omega(p,q)\,, \tag{26a}$$

$$\begin{aligned} a(p_0,q_0) &= \frac{1}{2^n \pi^{n-3}} \int_0^\infty d\bar{s}\, \bar{s}^{n-5} \\ &\times \Big[\tfrac{1}{2}\Big(n(\bar{s}-p_0) + (n(\bar{s}+p_0) - 2n(\bar{s}) \Big) \\ &+ n(q_0)\Big(n(\bar{s}+q_0) + n(\bar{s}-q_0) - 2n(\bar{s}) \Big) \\ &+ n(p_0+q_0)\Big(n(\bar{s}+p_0) + n(\bar{s}-p_0) \\ &- n(\bar{s}+q_0) - n(\bar{s}-q_0) \Big) \Big]\,, \end{aligned} \tag{26b}$$

$$\begin{aligned} \omega(p,q) &= \int d\Omega_{n-1}\, \delta(q\cdot V_+)\, \delta(p\cdot V_+) \\ &= \int d\Omega_{n-1}\, \delta(q\cdot V_-)\, \delta(p\cdot V_-)\,. \end{aligned} \tag{26c}$$

The angular integral (26c) is identical with the one found by Taylor [11] for the spectral density of the 3-gluon vertex in hot QCD. For $n = 6$ space-time dimensions (for which the theory is renormalizable) the radial integral (26b) is easily evaluated with the help of

$$\begin{aligned} I(a) &= \int_0^\infty d\bar{s}\, \bar{s} \left(\frac{1}{e^{\bar{s}+a}-1} + \frac{1}{e^{\bar{s}-a}-1} \right) \\ &= \int_0^a (x-a)\, dx + 2\int_0^\infty \frac{x\, dx}{e^x - 1} = \frac{\pi^2}{3} - \frac{a^2}{2}\,. \end{aligned} \tag{27}$$

We find

$$\begin{aligned} a(p_0,q_0) = -\frac{1}{2^7\pi^3} \Big[& p_0^2 \Big(\tfrac{1}{2} + n(p_0+q_0) \Big) \\ & + q_0^2 \Big(n(q_0) - n(q_0+p_0) \Big) \Big]\,. \end{aligned} \tag{28}$$

For $p_0, q_0 \ll T$ this goes to

$$a(p_0,q_0) \approx -\frac{1}{2^7\pi^3} p_0 T\,. \tag{29}$$

For $p_0 \sim \sqrt{g}\,T \ll T$ this is much smaller than the leading T^2-behaviour expected on dimensional grounds; this implies that the assumption that the loop integral is dominated by hard momenta $\bar{s} \sim T$ was wrong, and that in scalar ϕ^3 theory *there is no leading HTL contribution to the 3-point vertex*, in contrast to the case of gauge theories. Braaten-Pisarski resummation for ϕ^3 theory can thus be performed with bare 3-point vertices. A similar result was obtained in [26,27] for scalar QED. The existence of leading HTL contributions to vertices in QCD and fermionic QED can be traced back to the existence of Ward identities which connect vertex corrections with self energy corrections [9,27,28].

Before proceeding to a more accurate evaluation of the spectral density for vanishing spatial external momenta, let us shortly comment on the other spectral density which is obtained from

$$\rho_2' = -\frac{1}{n(q_0)} \mathrm{Im} G_{212}\,. \tag{30}$$

By inspection of the corresponding labelling of the diagram in Fig. 1 one observes that $G_{212}(p,q,-p-q)$ is obtained from $G_{122}(p,q,-p-q)$ by exchanging the legs with the external momenta p and q and routing the internal momentum s in the opposite direction. This yields the identity[2]

$$\rho_2'(p,q) = \rho_1'(q,p)\,. \tag{31}$$

For the HTL contributions to the corresponding loop integrals we thus obtain

$$\rho_1^{\mathrm{HTL}}(p,q) = p_0\, \rho_{\mathrm{HTL}}(p,q)\,, \tag{32a}$$

$$\rho_2^{\mathrm{HTL}}(p,q) = q_0 \rho_{\mathrm{HTL}}(p,q)\,, \tag{32b}$$

$$\rho_{\mathrm{HTL}}(p,q) \approx \frac{g^3 T}{2^7\pi^3} \int d\Omega_5\, \delta(p\cdot V_+)\, \delta(q\cdot V_+)\,. \tag{32c}$$

Up to a trivial external momentum factor the two spectral densities are thus equal to each other in the HTL limit. This agrees with the observation by Taylor [11] that in QCD in HTL approximation the two independent spectral densities for the 3-gluon vertex degenerate.

The result (29) shows that the loop integral is *not* dominated by hard momenta of order T, contrary to the assumption under which the integral was evaluated. This means that the HTL result for the spectral density is not reliable as an order of magnitude estimate, not even for power counting in the coupling constant g. In general a better estimate is difficult to obtain because for small loop momenta the radial and angular integrations cannot be decoupled. Things simplify, however, for vanishing external spatial momenta, $\boldsymbol{q} = \boldsymbol{p} = 0$. In this limit we find

$$\begin{aligned} & A(p_0,q_0) \\ & = n(p_0) \int \frac{d^{n-1}s}{(2\pi)^{n-3}}\, \frac{1}{2E_s}\, \mathrm{sgn}(E_s+q_0) \end{aligned}$$

[2] Note that such an identity is not expected to hold for the 3-point vertex in Yukawa theory or in other theories where different types of fields are attached to the vertex.

$$\times \,\mathrm{sgn}(E_s - p_0)\, \frac{\delta(2E_s + q_0)\,\delta(2E_s - p_0)}{|p_0||q_0|}$$
$$\times \Big[\tfrac{1}{2}\big(n(E_s - p_0) - n(E_s)\big)$$
$$- n(p_0+q_0)\big(n(E_s+q_0) - n(E_s-p_0)\big)$$
$$+ n(q_0)\big(n(E_s+q_0) - n(E_s)\big)\Big]\,, \tag{33a}$$

$$B(p_0,q_0)$$
$$= n(p_0)\int \frac{d^{n-1}s}{(2\pi)^{n-3}}\,\frac{1}{2E_s}\,\mathrm{sgn}(E_s - q_0)$$
$$\times \,\mathrm{sgn}(E_s + p_0)\, \frac{\delta(2E_s - q_0)\,\delta(2E_s + p_0)}{|p_0||q_0|}$$
$$\times \Big[\tfrac{1}{2}\big(n(E_s + p_0) - n(E_s)\big)$$
$$- n(p_0+q_0)\big(n(E_s-q_0) - n(E_s+p_0)\big)$$
$$+ n(q_0)\big(n(E_s-q_0) - n(E_s)\big)\Big]\,, \tag{33b}$$

Due to the δ-functions p_0 and q_0 must have the same magnitude and opposite sign, and A contributes only for $p_0 = -q_0 > 2m$ while B contributes for $p_0 = -q_0 < -2m$. The angular integrations are now trivial, and the radial integration is easily performed using the δ-functions. The final result, to leading order in the small ratios p_0/T, q_0/T, is

$$\rho'_1(p_0,q_0;\boldsymbol{p}=\boldsymbol{q}=0)$$
$$\approx -\frac{g^3}{12\pi}\,\frac{(p_0^2-4m^2)^{3/2}}{p_0^2 q_0^2}T^2$$
$$\times \Big[\theta(p_0-2m) - \theta(-p_0-2m)\Big]\,\delta(p_0+q_0)\,. \tag{34}$$

Using (31) one obtains for the other spectral density

$$\rho'_2(p_0,q_0;\boldsymbol{p}=\boldsymbol{q}=0) = \rho'_1(q_0,p_0;\boldsymbol{q}=\boldsymbol{p}=0)$$
$$= -\rho'_1(p_0,q_0;\boldsymbol{p}=\boldsymbol{q}=0)\,. \tag{35}$$

For $p_0 \sim q_0 \sim \sqrt{g}\,T$ power counting shows that these spectral densities are of order g^2. Inserting them into the spectral representations (15) and evaluating the latter via residue calculus it is easy to see [29] that the retarded 1-loop 1PI vertex functions at zero external spatial momenta are of the same order, i.e. one order of g down relative to the tree-level vertex. This reconfirms the above conclusion that in scalar ϕ_6^3 theory no vertex resummation is necessary.

IV Conclusions

In the CTP approach we have derived a set of useful relations among the eight thermal components of the 3-point vertex function many of which we have not previously seen in the literature in this form. They simplify formal manipulations in the real-time formulation of finite temperature field theory. With their help we have found an alternative derivation of spectral representations, in terms of two independent spectral densities, for the various thermal components of the real-time 3-point vertex at finite temperature; they appear simpler than those given in the literature before.

We then proceeded to an evaluation of these two spectral densities for the 3-point vertex in hot ϕ^3 theory in 5+1 dimensions, in the 1-loop approximation for soft external momenta $p, q \sim \sqrt{g}\,T$. This scale is set by the value of the (resummed) scalar mass m which is required to render the vacuum in ϕ_6^3 theory perturbatively stable. We found that, contrary to the case of the 3-gluon vertex in QCD, the loop integral for the spectral density for the scalar 3-point vertex is not dominated by hard momenta of order T, and the popular HTL approximation which decouples the radial and angular integrals produces an unreliable result. On the other hand, this means that even for soft external momenta the 1-loop vertex is of lower order than the tree-level contribution, and no vertex resummation is necessary in the Braaten-Pisarski high-temperature resummation scheme. An explicit evaluation of the 1-loop spectral densities for the 3-point vertex at vanishing external spatial momenta yields a result which is of order g^2, one power of g (but not two powers of g as in naive perturbation theory) below the tree-level vertex.

We also showed that the two independent spectral densities for the 3-point vertex in ϕ^3 theory are very closely related by the simple symmetry relation (31). At vanishing external spatial momenta they become, up to a sign, equal to each other. This should be compared with the finding of Taylor [11] that in QCD in HTL approximation the two spectral densities for the 3-gluon vertex become identical.

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Appendix A: derivation of spectral representations

A.1 Connected three-point functions

We start from the explicit expressions for the retarded connected 3-point vertex functions in ϕ^3 theory:

$$\Gamma_R = \theta_{23}\theta_{31}\langle[[\phi_2,\phi_3],\phi_1]\rangle + \theta_{21}\theta_{13}\langle[[\phi_2,\phi_1],\phi_3]\rangle\,, \tag{A1a}$$
$$\Gamma_{Ri} = \theta_{12}\theta_{23}\langle[[\phi_1,\phi_2],\phi_3]\rangle + \theta_{13}\theta_{32}\langle[[\phi_1,\phi_3],\phi_2]\rangle\,, \tag{A1b}$$
$$\Gamma_{Ro} = \theta_{32}\theta_{21}\langle[[\phi_3,\phi_2],\phi_1]\rangle + \theta_{31}\theta_{12}\langle[[\phi_3,\phi_1],\phi_2]\rangle\,. \tag{A1c}$$

(i) We begin with Γ_{Ro}. Inserting the identities

$$\theta_{31}\Gamma_{Ro} = \Gamma_{Ro} = \theta_{32}\Gamma_{Ro} \tag{A2}$$

into

$$\Gamma_{Ro} = (\theta_{21} + \theta_{12})\Gamma_{Ro} \quad \text{(A3)}$$

one obtains

$$\Gamma_{Ro} = \theta_{31}\theta_{12}\Gamma_{Ro} + \theta_{32}\theta_{21}\Gamma_{Ro} \,. \quad \text{(A4)}$$

Subtraction of the identities

$$\theta_{31}\theta_{12}\Gamma_R = 0 = \theta_{32}\theta_{21}\Gamma_{Ri} \,, \quad \text{(A5)}$$

which result from conflicting θ-functions, then gives

$$\begin{aligned}\Gamma_{Ro} &= \theta_{32}\theta_{21}(\Gamma_{Ro} - \Gamma_{Ri}) + \theta_{31}\theta_{12}(\Gamma_{Ro} - \Gamma_R) \\ &= \theta_{32}\theta_{21}(\Gamma_{221} + \Gamma_{112} - \Gamma_{211} - \Gamma_{122}) \\ &\quad + \theta_{31}\theta_{12}(\Gamma_{221} + \Gamma_{112} - \Gamma_{121} - \Gamma_{212}) \,. \quad \text{(A6)}\end{aligned}$$

With the help of (10a) and (7a) this is transformed into

$$\begin{aligned}\Gamma_{Ro} &= \theta_{32}\theta_{21}(\Gamma_{222} + \Gamma_{111} - \Gamma_{211} - \Gamma_{122}) \\ &\quad + \theta_{31}\theta_{12}(\Gamma_{222} + \Gamma_{111} - \Gamma_{121} - \Gamma_{212}) \\ &= \theta_{32}\theta_{21}(\tilde{\Gamma}_{111} + \tilde{\Gamma}_{222} - \Gamma_{211} - \Gamma_{122}) \\ &\quad + \theta_{31}\theta_{12}(\tilde{\Gamma}_{111} + \tilde{\Gamma}_{222} - \Gamma_{121} - \Gamma_{212}) \,. \quad \text{(A7)}\end{aligned}$$

Using also (11b,c) one finally gets

$$\begin{aligned}\Gamma_{Ro} &= \theta_{32}\theta_{21}\big(\tilde{\Gamma}_{122} + \tilde{\Gamma}_{211} - (\Gamma_{122} + \Gamma_{211})\big) \\ &\quad + \theta_{31}\theta_{12}\big(\tilde{\Gamma}_{121} + \tilde{\Gamma}_{212} - (\Gamma_{121} + \Gamma_{212})\big) \,. \quad \text{(A8)}\end{aligned}$$

(ii) For Γ_R one proceeds similarly. One writes

$$\Gamma_R = (\theta_{31} + \theta_{13})\Gamma_R = \theta_{31}(\theta_{23}\Gamma_R) + \theta_{31}(\theta_{21}\Gamma_R) \quad \text{(A9)}$$

and subtracts the identities

$$\theta_{21}\theta_{13}\Gamma_{Ro} = 0 = \theta_{23}\theta_{31}\Gamma_{Ri} \,. \quad \text{(A10)}$$

This yields

$$\begin{aligned}\Gamma_R &= \theta_{21}\theta_{13}\big(\Gamma_{121} + \Gamma_{212} - (\Gamma_{112} + \Gamma_{221})\big) \\ &\quad + \theta_{23}\theta_{31}\big(\Gamma_{212} + \Gamma_{121} - (\Gamma_{211} + \Gamma_{122})\big) \,. \quad \text{(A11)}\end{aligned}$$

Using (10b) and (11a,c) this is then transformed into

$$\begin{aligned}\Gamma_R &= \theta_{21}\theta_{13}\big(\tilde{\Gamma}_{112} + \tilde{\Gamma}_{221} - (\Gamma_{112} + \Gamma_{221})\big) \\ &\quad + \theta_{23}\theta_{31}\big(\tilde{\Gamma}_{211} + \tilde{\Gamma}_{122} - (\Gamma_{211} + \Gamma_{122})\big) \,. \quad \text{(A12)}\end{aligned}$$

(iii) Finally, Γ_{Ri} is reexpressed by writing

$$\begin{aligned}\Gamma_{Ri} &= (\theta_{23} + \theta_{32})\Gamma_{Ri} \\ &= \theta_{23}(\theta_{12}\Gamma_{Ri}) + \theta_{32}(\theta_{13}\Gamma_{Ri}) \quad \text{(A13)}\end{aligned}$$

and subtracting the identities

$$\theta_{12}\theta_{23}\Gamma_{Ro} = 0 = \theta_{13}\theta_{32}\Gamma_R \,. \quad \text{(A14)}$$

This yields

$$\begin{aligned}\Gamma_{Ri} &= \theta_{12}\theta_{32}\big(\Gamma_{122} + \Gamma_{211} - (\Gamma_{112} + \Gamma_{221})\big) \\ &\quad + \theta_{13}\theta_{32}\big(\Gamma_{122} + \Gamma_{211} - (\Gamma_{121} + \Gamma_{212})\big) \,. \quad \text{(A15)}\end{aligned}$$

Using (10c) and (11a,b) this is transformed into

$$\begin{aligned}\Gamma_{Ri} &= \theta_{12}\theta_{23}\big(\tilde{\Gamma}_{112} + \tilde{\Gamma}_{221} - (\Gamma_{112} + \Gamma_{221})\big) \\ &\quad + \theta_{13}\theta_{32}\big(\tilde{\Gamma}_{121} + \tilde{\Gamma}_{212} - (\Gamma_{121} + \Gamma_{212})\big) \,. \quad \text{(A16)}\end{aligned}$$

(iv) We can summarize these results in coordinate space as follows:

$$\begin{aligned}\Gamma_R &= \theta_{21}\theta_{13}\bar{\rho}_3 + \theta_{23}\theta_{31}\bar{\rho}_1 \,, \quad &\text{(A17a)} \\ \Gamma_{Ri} &= \theta_{12}\theta_{23}\bar{\rho}_3 + \theta_{13}\theta_{32}\bar{\rho}_2 \,, \quad &\text{(A17b)} \\ \Gamma_{Ro} &= \theta_{32}\theta_{21}\bar{\rho}_1 + \theta_{31}\theta_{12}\bar{\rho}_2 \,, \quad &\text{(A17c)}\end{aligned}$$

where

$$\begin{aligned}\bar{\rho}_1 &= \tilde{\Gamma}_{122} + \tilde{\Gamma}_{211} - (\Gamma_{122} + \Gamma_{211}) \,, \quad &\text{(A18a)} \\ \bar{\rho}_2 &= \tilde{\Gamma}_{121} + \tilde{\Gamma}_{212} - (\Gamma_{121} + \Gamma_{212}) \,, \quad &\text{(A18b)} \\ \bar{\rho}_3 &= \tilde{\Gamma}_{112} + \tilde{\Gamma}_{221} - (\Gamma_{112} + \Gamma_{221}) \,. \quad &\text{(A18c)}\end{aligned}$$

(v) In momentum space tilde conjugation reduces to complex conjugation (see (7)), and the last three equations correspondingly reduce to

$$\begin{aligned}\bar{\rho}_1 &= -2i\mathrm{Im}\,(\Gamma_{122} + \Gamma_{211}) \,, \quad &\text{(A19a)} \\ \bar{\rho}_2 &= -2i\mathrm{Im}\,(\Gamma_{121} + \Gamma_{212}) \,, \quad &\text{(A19b)} \\ \bar{\rho}_3 &= -2i\mathrm{Im}\,(\Gamma_{112} + \Gamma_{221}) \,. \quad &\text{(A19c)}\end{aligned}$$

Using the Fourier integral representation of the θ function

$$\theta_{ij} = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} d\Omega \frac{e^{-i\Omega(t_i - t_j)}}{\Omega + i\epsilon} \,, \quad \text{(A20)}$$

it is then straightforward to derive the following spectral integral representations in momentum space:

$$\begin{aligned}&\Gamma_R(\omega_1, \omega_2, \omega_3) \\ &\quad = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty} \frac{d\Omega_1 d\Omega_2}{\omega_2 - \Omega_2 + i\epsilon} \\ &\quad \times \left(\frac{\rho_1(\Omega_1, \Omega_2, \Omega_3)}{\omega_1 - \Omega_1 - i\epsilon} + \frac{\rho_3(\Omega_1, \Omega_2, \Omega_3)}{\omega_3 - \Omega_3 - i\epsilon}\right) \,, \quad \text{(A21a)} \\ &\Gamma_{Ri}(\omega_1, \omega_2, \omega_3) \\ &\quad = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty} \frac{d\Omega_1 d\Omega_2}{\omega_1 - \Omega_1 + i\epsilon} \\ &\quad \times \left(\frac{\rho_2(\Omega_1, \Omega_2, \Omega_3)}{\omega_2 - \Omega_2 - i\epsilon} + \frac{\rho_3(\Omega_1, \Omega_2, \Omega_3)}{\omega_3 - \Omega_3 - i\epsilon}\right) \,, \quad \text{(A21b)} \\ &\Gamma_{Ro}(\omega_1, \omega_2, \omega_3) \\ &\quad = \frac{-i}{2\pi^2}\int_{-\infty}^{\infty} \frac{d\Omega_1 d\Omega_2}{\omega_3 - \Omega_3 + i\epsilon} \\ &\quad \times \left(\frac{\rho_1(\Omega_1, \Omega_2, \Omega_3)}{\omega_1 - \Omega_1 - i\epsilon} + \frac{\rho_2(\Omega_1, \Omega_2, \Omega_3)}{\omega_2 - \Omega_2 - i\epsilon}\right) \,, \quad \text{(A21c)}\end{aligned}$$

where $\omega_1 + \omega_2 + \omega_3 = \Omega_1 + \Omega_2 + \Omega_3 = 0$, and we used the new notation $\bar{\rho}_k = -2i\rho_k$ where (according to (A19)) the ρ_k are real functions. The spatial momenta on both sides of the equations are equal.

(vi) In [18] it was shown that only two of the spectral densities ρ_i are independent. This is consistent with the results derived here after realizing, by again using (10) and (11), that

$$\theta_{12}\theta_{23}(\bar\rho_1 - \bar\rho_2 + \bar\rho_3) = 0\,, \qquad \text{(A22a)}$$
$$\theta_{21}\theta_{13}(\bar\rho_2 - \bar\rho_1 + \bar\rho_3) = 0\,, \qquad \text{(A22b)}$$

such that one may substitute $\rho_3 = \rho_1 - \rho_2$ in Γ_R and $\rho_3 = \rho_2 - \rho_1$ in Γ_{Ri} (please note the opposite sign in the two cases!). This then yields (13).

A.2 1PI three-point functions

The 1PI or truncated vertex functions $G_{abc}(k_1,k_2,k_3)$ are obtained from the connected vertex functions Γ_{abc} by truncating the three external propagators:

$$G_{abc}(k_1,k_2,k_3) = \frac{1}{i^3} D^{-1}_{aa'}(k_1) D^{-1}_{bb'}(k_2) \times D^{-1}_{cc'}(k_3)\Gamma_{a'b'c'}(k_1,k_2,k_3)\,. \qquad \text{(A23)}$$

They satisfy the identity

$$\sum_{a,b,c=1}^{2} G_{abc} = 0\,, \qquad \text{(A24)}$$

and the three retarded 1PI vertices are given by [7,18]

$$G_R = G_{111} + G_{112} + G_{211} + G_{212}\,, \qquad \text{(A25a)}$$
$$G_{Ri} = G_{111} + G_{112} + G_{121} + G_{122}\,, \qquad \text{(A25b)}$$
$$G_{Ro} = G_{111} + G_{121} + G_{211} + G_{221}\,. \qquad \text{(A25c)}$$

These relations differ from (6) and (12) only by sign factors $(-1)^{a+b+c-3}$. In momentum space we have instead of (7) [20]

$$\tilde G_{111}(k_1,k_2,k_3) = -G^*_{111}(k_1,k_2,k_3) = G_{222}(k_1,k_2,k_3)\,, \qquad \text{(A26a)}$$
$$\tilde G_{121}(k_1,k_2,k_3) = -G^*_{121}(k_1,k_2,k_3) = e^{\beta\omega_2}\, G_{212}(k_1,k_2,k_3)\,, \qquad \text{(A26b)}$$
$$\tilde G_{211}(k_1,k_2,k_3) = -G^*_{211}(k_1,k_2,k_3) = e^{\beta\omega_1}\, G_{122}(k_1,k_2,k_3)\,, \qquad \text{(A26c)}$$
$$\tilde G_{112}(k_1,k_2,k_3) = -G^*_{112}(k_1,k_2,k_3) = e^{\beta\omega_3}\, G_{221}(k_1,k_2,k_3)\,, \qquad \text{(A26d)}$$

where "tilde conjugation" is defined in the same way as for the connected vertex.

From identities like $\theta_{32}\theta_{21}(G_R + G_{Ri}) = 0$ involving conflicting θ-functions one can derive largest and smallest time equations for the 1PI vertices similar to those derived in Sect. II.2 for the connected vertices. One finds that the sign factor $(-1)^{a+b+c-3}$ simply carries over, changing all relative minus signs in (9)-(11) into plus signs. For example, (9) turns into

$$\theta_{32}\theta_{21}(G_{111} + G_{112}) = 0\,. \qquad \text{(A27)}$$

By following the same procedure as for the connected functions in Appendix A.1 one obtains for the truncated functions the following relations in coordinate space:

$$G_R = \theta_{21}\theta_{13}\bar\rho'_3 + \theta_{23}\theta_{31}\bar\rho'_1\,, \qquad \text{(A28a)}$$
$$G_{Ri} = \theta_{12}\theta_{23}\bar\rho'_3 + \theta_{13}\theta_{32}\bar\rho'_2\,, \qquad \text{(A28b)}$$
$$G_{Ro} = \theta_{32}\theta_{21}\bar\rho'_1 + \theta_{31}\theta_{12}\bar\rho'_2\,, \qquad \text{(A28c)}$$

where

$$\bar\rho'_1 = \tilde G_{211} - \tilde G_{122} + G_{211} - G_{122}\,, \qquad \text{(A29a)}$$
$$\bar\rho'_2 = \tilde G_{121} - \tilde G_{212} + G_{121} - G_{212}\,, \qquad \text{(A29b)}$$
$$\rho'_3 = \tilde G_{112} - \tilde G_{221} + G_{112} - G_{221}\,. \qquad \text{(A29c)}$$

In momentum space, by making use of (A26), the last three equations reduce to

$$\bar\rho'_1 = -2i\,\mathrm{Im}\,(G_{122} - G_{211})\,, \qquad \text{(A30a)}$$
$$\bar\rho'_2 = -2i\,\mathrm{Im}\,(G_{212} - G_{212})\,, \qquad \text{(A30b)}$$
$$\bar\rho'_3 = -2i\,\mathrm{Im}\,(G_{221} - G_{221})\,. \qquad \text{(A30c)}$$

Using further (A24), the largest and smallest time equations, and (A26) for the truncated functions one shows that

$$\theta_{12}\theta_{23}(\bar\rho'_1 - \bar\rho'_2 + \bar\rho'_3) = 0\,, \qquad \text{(A31a)}$$
$$\theta_{21}\theta_{13}(\bar\rho'_2 - \bar\rho'_1 + \bar\rho'_3) = 0\,. \qquad \text{(A31b)}$$

Inserting these into (A28) and transforming to momentum space one obtains the spectral integrals for the truncated vertex functions given in (15). Please note that, up to the different definition of the spectral densities, they are formally identical with the spectral representations (13) for the corresponding connected vertex functions.

Appendix B: symmetries of the three-point spectral densities

In addition to (31), which holds only for 3-point vertices with three identical external legs, there are some useful other symmetries for the 3-point spectral densities. Inserting the propagators (2) into (18) and using twice the relation (20) one obtains, without any further manipulations, the expression

$$\begin{aligned}\rho'_1(p,q) &= g^3 \int \frac{d^n s}{(2\pi)^{n-3}} \mathrm{sgn}(s_0)\,\mathrm{sgn}(s_0+q_0)\,\mathrm{sgn}(s_0-p_0) \\ &\quad\times \delta(s^2-m^2)\,\delta\big((s+q)^2-m^2\big)\,\delta\big((s-p)^2-m^2\big) \\ &\quad\times \Big[\tfrac12\Big(n(s_0)-n(s_0-p_0)\Big) \\ &\quad + n(q_0)\Big(n(s_0)-n(s_0+q_0)\Big) \\ &\quad + n(p_0+q_0)\Big(n(s_0+q_0)-n(s_0-p_0)\Big)\Big]\,. \end{aligned} \qquad \text{(B1)}$$

By reversing the sign of the spatial integration variable, $\boldsymbol{s} \mapsto -\boldsymbol{s}$, one immediately reads off

$$\rho'_1(p_0,\boldsymbol{p};q_0,\boldsymbol{q}) = \rho'_1(p_0,-\boldsymbol{p};q_0,-\boldsymbol{q})\,. \qquad \text{(B2)}$$

With the additional help of the identity $n(-x) = -\big(1 + n(x)\big)$ one shows similarly that

$$\rho_1'(-p,-q) = -\rho_1'(p,q)\,. \tag{B3}$$

The combination of these two identities gives

$$\rho_1'(-p_0,\boldsymbol{p};-q_0,\boldsymbol{q}) = -\rho_1'(p_0,\boldsymbol{p};q_0,\boldsymbol{q})\,, \tag{B4}$$

i.e. the spectral density is odd under simultaneous sign change of both frequencies. This generalizes a similar condition for the spectral density for the single-particle propagator. The above symmetry relations are consistent with (4) and (5) in [24].

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